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LETTER TO THE EDITOR

An exactly soluble nuclear model with $SU_q(3) \supset SU_q(2) \supset SO_q(2)$ symmetry

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Abstract. The q-deformed version of a two-dimensional toy interacting boson model (IBM) with the symmetry $SU_q(3) \supset SU_q(2) \supset SO_q(2)$ is constructed. Energy spectra and transition matrix elements are calculated, the latter being found to be much more sensitive to q-deformation than the former. Arguments in favour of the q-generalization of the full IBM are given.

Quantum algebras [1–4], which from the mathematical point of view are Hopf algebras as pointed out in [3], are recently attracting much attention in physics, especially after the introduction of the q-deformed harmonic oscillator [5–7]. Initially used in the quantum inverse scattering problem for solving the Yang–Baxter equation [8, 9], they have been subsequently used in conformal field theories [10, 11], in the description of spin chains [12, 13], as well as in the description of squeezed states [14]. In nuclear physics the q-rotator having the symmetry $SU_q(2)$ has been successfully used for the description of rotational spectra of deformed [15, 16] and superdeformed [17] nuclei, and its equivalence to the variable moment of inertia (VMI) model has been demonstrated [16]. The deformation parameter τ^2 (with $q = e^{i\tau}$) of the $SU_q(2)$ model has been found [16] to correspond to the softness parameter of the VMI model, thus indicating that the q-deformation of the ususal SU(2) algebra is physically well motivated. Applications of $SU_q(2)$ and $SU_q(1,1)$ symmetries in rotational [18] and vibrational [19, 20] spectra of diatomic molecules also exist. All these applications, however, are confined to the relatively simple $SU_q(2)$ and $SU_q(1,1)$ algebras. On the other hand, algebraic models using more complicated algebras have been in recent years very successful in describing low-lying collective spectra of medium and heavy mass nuclei away from closed shells, in which the use of the shell model is not yet possible. The most widely used algebraic model of nuclear collectivity is the interacting boson model (IBM) ([21], for recent overviews see [22, 23]), in the simplest version of which low-lying collective nuclear spectra are described in terms of s ($J=0$) and d ($J=2$) bosons, which are supposed to be correlated fermion pairs. The symmetry of the simplest version of the model is U(6), which contains U(5) (vibrational), SU(3) (rotational)

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and O(6) (γ-unstable) chains of subalgebras. A simplified version of the model, having the SU(3) symmetry with SU(2) and SO(3) chains of subalgebras also exists [24]. It can be considered as a toy model for two-dimensional nuclei, but it is very useful in demonstrating the basic techniques used in the full IBM.

In the present letter we attempt the first application of quantum algebras beyond $SU_q(2)$ or $SU_q(1,1)$ in nuclear physics, by constructing a q-generalization of the two-dimensional toy IBM. The model has the $SU_q(3)$ symmetry, which possesses an $SU_q(2)$ subalgebra. Its construction gives several hints about the possible q-generalization of the full IBM and its possible usefulness. A source of motivation towards constructing the q-generalization of the IBM is the recent proof [25] that the description of fermion pairs of zero angular momentum ($J=0$) in a single-j shell in terms of q-bosons is much simpler than their description in terms of usual bosons, which is an indication that q-bosons might be more appropriate than usual bosons for the description of correlated fermion pairs.

In the classical version of the toy IBM [24] one introduced bosons with angular momentum $m=0, \pm 2$, represented by the creation (annihilation) operators a_0^+, a_+^+, a_-^+ (a_0, a_+, a_-). They satisfy usual boson commutation relations

$$[a_i, a_j^+]=\delta_{ij} \qquad [a_i, a_j]=[a_i^+, a_j^+]=0. \tag{1}$$

The nine bilinear operators

$$\Lambda_{ij}=a_i^+ a_j \tag{2}$$

satisfy then the commutation relations

$$[\Lambda_{ij}, \Lambda_{kl}]=\delta_{jk}\Lambda_{il}-\delta_{il}\Lambda_{jk} \tag{3}$$

which are the standard U(3) commutation relations. The total number of bosons

$$N=\Sigma_i \Lambda_{ii}=a_0^+ a_0+a_+^+ a_+ + a_-^+ a_- \tag{4}$$

is kept constant.

In the quantum case one has the $U_q(3)$ commutation relations given in table 1 [26], where A_{ij} are the generators of $U_q(3)$ and the q-commutator is defined as

$$[A, B]_q=AB-qBA. \tag{5}$$

In order to obtain a realization of $U_q(3)$ in terms of q-bosons, satisfying the commutation relations [5-7]

$$a_i a_i^+ - q a_i^+ a_i = q^{-N_i} \qquad a_i a_i^+ - q^{-1} a_i^+ a_i = q^N \tag{6}$$

and satisfying

$$a_i^+ a_i=[N_i] \qquad a_i a_i^+=[N_i+1] \tag{7}$$

where

$$[x]=\frac{q^x-q^{-x}}{q-q^{-1}} \tag{8}$$

is the definition of q-numbers, one starts with [27]

$$A_{12}=a_1^+ a_2 \qquad A_{21}=a_2^+ a_1 \tag{9}$$

$$A_{23}=a_2^+ a_3 \qquad A_{32}=a_3^+ a_2. \tag{10}$$

Table 1. $U_q(3)$ commutation relations [26], given in the form $[A, B]_a = C$. A is given in the first column, B in the first row. C is given at the intersection of the row containing A with the column containing B. a, when different from 1, follows C, enclosed in parentheses.

	A_{11}	A_{22}	A_{33}	A_{12}	A_{23}	A_{13}
A_{11}	0	0	0	A_{12}	0	A_{13}
A_{22}	0	0	0	$-A_{12}$	A_{23}	0
A_{33}	0	0	0	0	$-A_{23}$	A_{13}
A_{12}	$-A_{12}$	A_{12}	0	0	$A_{13}\,(q)$	$0\,(q^{-1})$
A_{23}	0	$-A_{23}$	A_{23}	$-q^{-1}A_{13}(q^{-1})$	0	$0\,(q)$
A_{13}	$-A_{13}$	0	A_{13}	$0\,(q)$	$0\,(q^{-1})$	0
A_{21}	A_{21}	$-A_{21}$	0	$-[A_{11}-A_{22}]$	0	$A_{23}q^{A_{11}-A_{22}}$
A_{32}	0	A_{32}	$-A_{32}$	0	$-[A_{22}-A_{33}]$	$-q^{-A_{22}+A_{33}}A_{12}$
A_{31}	A_{31}	0	$-A_{31}$	$-q^{-A_{11}+A_{22}}A_{32}$	$-A_{21}q^{A_{22}-A_{33}}$	$-[A_{11}-A_{33}]$

	A_{21}	A_{32}	A_{31}
A_{11}	$-A_{21}$	0	$-A_{31}$
A_{22}	A_{21}	$-A_{32}$	0
A_{33}	0	A_{32}	A_{31}
A_{12}	$[A_{11}-A_{22}]$	0	$-q^{-A_{11}+A_{22}}A_{32}$
A_{23}	0	$[A_{22}-A_{33}]$	$A_{21}q^{A_{22}-A_{33}}$
A_{13}	$-A_{23}q^{A_{11}-A_{22}}$	$q^{-A_{22}+A_{33}}A_{12}$	$[A_{11}-A_{33}]$
A_{21}	0	$-qA_{31}(q)$	$0\,(q^{-1})$
A_{32}	$A_{31}(q^{-1})$	0	$0\,(q)$
A_{31}	$0(q)$	$0(q^{-1})$	0

One can easily verify that the $U_q(3)$ commutation relations involving these generators are satisfied. For example, one has

$$[A_{12}, A_{21}] = [N_1 - N_2] \tag{11}$$

$$[A_{23}, A_{32}] = [N_2 - N_3] \tag{12}$$

using the identity

$$[N_i][N_j+1] - [N_j][N_i+1] = [N_i - N_j] \tag{13}$$

and the identifications

$$N_1 = A_{11} \qquad N_2 = A_{22} \qquad N_3 = A_{33}. \tag{14}$$

One can now determine the boson realizations of A_{13} and A_{31} from other commutation relations, as follows

$$A_{13} = [A_{12}, A_{23}]_q = a_1^+ a_3 q^{-N_2} \tag{15}$$

$$A_{31} = [A_{32}, A_{21}]_{q^{-1}} = a_3^+ a_1 q^{N_2}. \tag{16}$$

Using (13) once more one can verify that the relation

$$[A_{13}, A_{31}] = [N_1 - N_3] \tag{17}$$

is fulfilled by the boson images of (15), (16). It is by now a straightforward task to verify that all commutation relations of table 1 are fulfilled by the boson images obtained above.

So far we have managed to write a boson realization of $U_q(3)$ in terms of three q-bosons, namely a_1, a_2, a_3. Omitting the generators involving one of the bosons, one is left with an $SU_q(2)$ subalgebra. Omitting the generators involving a_3, for example, one is left with A_{12}, A_{21}, N_1, N_2, which satisfy the $SU_q(2)$ commutation relations [5-7]

$$[J_+, J_-] = [2J_0] \qquad [J_0, J_\pm] = \pm J_\pm \tag{18}$$

where the identifications

$$J_+ = A_{12} \qquad J_- = A_{21} \qquad J_0 = \tfrac{1}{2}(N_1 - N_2) \tag{19}$$

have been made. J_0 alone forms an $SO_q(2)$ subalgebra. Therefore the relevant chain of subalgebras is

$$SU_q(3) \supset SU_q(2) \supset SO_q(2). \tag{20}$$

The second-order Casimir operator of $SU_q(2)$ is known to have the form [5-7]

$$C_2(SU_q(2)) = J^2 = J_-J_+ + [J_0][J_0+1]. \tag{21}$$

Substituting the above expressions for the generators one finds

$$C_2(SU_q(2)) = \left[\frac{N_1+N_2}{2}\right]\left[\frac{N_1+N_2}{2}+1\right]. \tag{22}$$

All of the above equations go to their classical counterparts by allowing $q \to 1$, for which $[x] \to x$, i.e. q-numbers become usual numbers. In the classical case [24] out of the three bosons (a_0, a_+, a_-) forming SU(3), one chooses to leave out a_0, the boson with zero angular momentum, in order to be left with the SU(2) subalgebra formed by a_+ and a_-, the two bosons of angular momentum 2. The choice of the $SU_q(2)$ subalgebra made above is then consistent with the following correspondence between classical bosons and q-bosons

$$a_+ \to a_1 \qquad a_- \to a_2 \qquad a_0 \to a_3. \tag{23}$$

(We have opted for using different indices for usual bosons and q-bosons in order to avoid confusion.)

In the classical case [24] the states of the system are characterized by the quantum numbers characterizing the irreducible representations (irreps) of the algebras appearing in the classical counterpart of the chain of equation (20). For SU(3) the total number of bosons N is used. For SU(2) and SO(2) one can use the eigenvalues of J^2 and J_0, or, equivalently, the eigenvalues of $a_+^\dagger a_+ + a_-^\dagger a_-$ and $L_3 = 4J_0$, for which we use the symbols n_d (the number of bosons with angular momentum 2) and M. Then the basis in the classical case can be written as [24]

$$|N, n_d, M\rangle = \frac{(a_0^\dagger)^{N-n_d}}{(N-n_d)!}\frac{(a_+^\dagger)^{n_d/2+M/4}}{(n_d/2+M/4)!}\frac{(a_-^\dagger)^{n_d/2-M/4}}{(n_d/2-M/4)!}|0\rangle. \tag{24}$$

In the quantum case for each oscillator one defines the basis as [5-7]

$$|N_i\rangle = \frac{(a_i^\dagger)^{N_i}}{[N_i]!}|0\rangle \tag{25}$$

where the q-factorial is defined as

$$[N]! = [N][N-1][N-2]\ldots[2][1]. \tag{26}$$

Then one has [5-7]

$$N_i|N_i\rangle = N_i|N_i\rangle \tag{27}$$

$$a_i^\dagger|N_i\rangle = \sqrt{[N_i+1]}\,|N_i+1\rangle \tag{28}$$

$$a_i|N_i\rangle = \sqrt{[N_i]}\,|N_i-1\rangle \tag{29}$$

so that the full basis in the q-deformed case is

$$|N, n_d, M\rangle_q = \frac{(a_3^+)^{N-n_d}}{[N-n_d]!} \frac{(a_1^+)^{n_d/2+M/4}}{[n_d/2+M/4]!} \frac{(a_2^+)^{n_d/2-M/4}}{[n_d/2-M/4]!}|0\rangle \tag{30}$$

where $N = N_1 + N_2 + N_3$ is the total number of bosons, $n_d = N_1 + N_2$ is the number of bosons with angular momentum 2, and M is the eigenvalue of $L = 4J_0$. n_d takes values from 0 up to N, while for a given value of n_d, M takes the values $\pm 2n_d$, $\pm 2(n_d - 1)$, $\pm 2(n_d - 2), \ldots, \pm 2$ or 0, depending on whether n_d is odd or even [24]. In this basis the eigenvalues of the second-order Casimir operator of $SU_q(2)$ are then

$$C_2(SU_q(2))|N, n_d, M\rangle_q = \left[\frac{n_d}{2}\right]\left[\frac{n_d}{2}+1\right]|N, n_d, M\rangle_q. \tag{31}$$

In the case of $N = 5$ one can easily see that the spectrum will be composed by the ground state band, consisting of states with $M = 0, 2, 4, 6, 8, 10$ and $n_d = M/2$, the first excited band with states characterized by $M = 0, 2, 4, 6$ and $n_d = M/2+2$, and the second excited band, containing sztes with $M = 0, 2$ and $n_d = M/2+4$.

In the case that the Hamiltonian has the $SU_q(2)$ dynamical symmetry, it can be written in terms of the Casimir operators of the chain (20). Then one has

$$H = E_0 + AC_2(SU_q(2)) + BC_2(SO_q(2)) \tag{32}$$

where E_0, A, B are constants. Its eigenvalues are

$$E = E_0 + A\left[\frac{n_d}{2}\right]\left[\frac{n_d}{2}+1\right] + BM^2. \tag{33}$$

Realistic nuclear spectra are characterized by strong electric quadrupole transitions among the levels of the same band, as well as by interband transitions. In the framework of the present toy model one can define, by analogy to the classical case [24], quadrupole transition operators

$$Q_+ = a_1^+ a_3 + a_3^+ a_2 \tag{34}$$

$$Q_- = a_2^+ a_3 + a_3^+ a_1. \tag{35}$$

In order to calculate transition matrix elements of these operators one only needs equations (28), (29), i.e. the action of the q-boson operators on the q-deformed basis. The selection rules, as in the classical case, are $\Delta M = \pm 2$, $\Delta n_d = \pm 1$, while the corresponding matrix elements are

$${}_q\langle N, n_d+1, M\pm 2|Q_\pm|N, n_d, M\rangle_q = \sqrt{[N-n_d]\left[\frac{n_d}{2} \pm \frac{M}{4} + 1\right]} \tag{36}$$

$${}_q\langle N, n_d-1, M\pm 2|Q_\pm|N, n_d, M\rangle_q = \sqrt{[N-n_d+1]\left[\frac{n_d}{2} \mp \frac{M}{4}\right]}. \tag{37}$$

From these equations it is clear that both intraband and interband transitions are possible.

In order to get a feeling of the qualitative changes in the spectrum and the transition matrix elements resulting from the q-deformation of the model, we make a simple calculation for a system of twenty bosons ($N = 20$). We distinguish two cases:

(i) q can be real ($q = e^\tau$, with τ real), in which case q-numbers can be written as

$$[x] = \frac{\sinh \tau x}{\sinh \tau}. \tag{38}$$

(ii) q can be a phase ($q = e^{i\tau}$, with τ real), in which case q-numbers can be put in the form

$$[x] = \frac{\sin \tau x}{\sin \tau}. \tag{39}$$

In order to isolate the effects of q-deformation in the spectrum, we consider a Hamiltonian (32) with $E_0 = 0$, $A = 1$, $B = 0$. As we have already remarked, the ground state band contains states characterized by $M = 2n_d$. Results for the lowest 10 members of the ground state band are reported in table 2 for the classical case ($\tau = 0$), as well as for the two q-deformed cases (q real, q a phase) for two different values of the deformation parameter ($\tau = 0.05$, 0.1). We remark that when q is real the spectrum is increasing more rapidly than in the classical case, while when q is a phase the spectrum increases more slowly than in the classical case. This is in agreement with the findings of the q-rotator model [15–18], which is equivalent to the VMI model for q being a phase, with τ having a well defined physical meaning (τ^2 is equivalent to the softness parameter of the VMI model [16]).

In table 3 we report for the same cases results for the transition matrix element

$${}_q\langle N, n_d + 1, M + 2|Q_+|N, n_d, M\rangle_q \tag{40}$$

which within the ground state band (where $M = 2n_d$) takes the form (see equation (36))

$${}_q\langle N, n_d + 1, 2n_d + 2|Q_+|N, n_d, 2n_d\rangle_q = \sqrt{[N - n_d][n_d + 1]}. \tag{41}$$

Results up to $n_d = 9$ are reported, since the matrix elements for n_d and $N - n_d - 1$ are equal, as is easily seen from (41). We remark that the transition matrix elements in the case that q is real increase more rapidly than in the classical case, while they increase less rapidly than the classical case when q is a phase. We also remark that transition matrix elements are much more sensitive to q-deformation than energy spectra. This is an interesting feature, showing that q-deformed algebraic models can be much more flexible in the description of transition probabilities than their classical counterparts.

Table 2. The lowest ten members of the ground state band of a system of twenty bosons ($N = 20$) described by the Hamiltonian (32) with $E_0 = 0$, $A = 1$, $B = 0$. Results for the classical case, as well as for the q-deformed cases with q real and q a phase are reported. See text for further discussion.

		$\tau = 0.05$		$\tau = 0.1$	
n_d	$\tau = 0$ classical	real	phase	real	phase
1	0.75	0.75	0.75	0.75	0.75
2	2.00	2.00	2.00	2.01	1.99
3	3.75	3.76	3.74	3.79	3.71
4	6.00	6.03	5.97	6.11	5.89
5	8.75	8.81	8.69	8.99	8.51
6	12.00	12.12	11.89	12.47	11.55
7	15.75	15.95	15.55	16.57	14.97
8	20.00	20.33	19.68	21.33	18.73
9	24.75	25.25	24.25	26.81	22.81
10	30.00	30.75	29.27	33.07	27.16

Table 3. The first ten transition matrix elements ${}_q\langle N, n_d+1, M+2|Q_+|N, n_d, M\rangle_q$ (equation (41)) among the levels of the ground state band (in which $M=2n_d$) of a system of twenty bosons ($N=20$). The classical results are reported, along with the q-deformed results for q real and q a phase. See text for further discussion.

		$\tau=0.05$		$\tau=0.1$	
n_d	$\tau=0$ classical	real	phase	real	phase
0	4.47	4.85	4.10	6.02	3.02
1	6.16	6.63	5.70	8.10	4.34
2	7.35	7.86	6.85	9.45	5.37
3	8.25	8.77	7.73	10.41	6.22
4	8.94	9.47	8.43	11.11	6.93
5	9.49	10.00	8.98	11.62	7.52
6	9.90	10.41	9.40	12.00	7.98
7	10.20	10.70	9.71	12.26	8.33
8	10.39	10.88	9.92	12.43	8.56
9	10.49	10.97	10.02	12.51	8.67

It should be noticed that the classical SU(3) toy model [24] considered here has, in addition to the above-mentioned and q-deformed SU(2) chain of subalgebras, an SO(3) chain. However, no $SO_q(3)$ subalgebra of the $SU_q(3)$ algebra has been obtained up to now [28].

In conclusion, we have developed the q-deformed generalization of a two-dimensional toy IBM model with $SU_q(3) \supset SU_q(2) \supset SO_q(2)$ symmetry. Spectra and transition matrix elements are influenced in different ways depending on whether q is real or a phase. Transition matrix elements are much more sensitive to q-deformation than energy levels. Therefore the development of q-deformed algebraic models applicable to realistic nuclei appears promising and will be pursued. In addition, the success of the q-rotator model [15–18] with $SU_q(2)$ symmetry in the description of rotational spectra suggests that its consequences for transition probabilities [29] should be examined. Work in these directions is in progress.

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